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ASYMPTOTIC SERIES ASSOCIATED WITH EPSTEIN ZETA-FUNCTIONS AND THEIR INTEGRAL TRANSFORMS

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1. INTRODUCTION

Throughout the following, $s=\sigma+it$ denotes the complex variable, and $z=x+iy$ the complex parameter in the upper-half plane. The main object of this article is the Epstein zeta-function (attached to the positive-definite quadratic form $|u+vz|^{2}$) defined by

$$\zeta_{\mathbf{Z}^{2}}(s;z)=\sum_{(m,n)\in\mathbf{Z}^{2}\setminus\{(0,0)\}}|m+nz|^{-2s} \qquad (\operatorname{Re}s>1), \tag{1.1}$$

and its meromorphic continuation over the whole s-plane (cf. [Si Chap. I]).

Let α, β be complex numbers which will be fixed later, and let $\Gamma(s)$ denote the gamma function. We introduce the Laplace-Mellin and the Riemann-Liouville (or the Erdélyi-Kober) transforms of $\zeta_{\mathbf{Z}^{2}}(s;x+iy)$ (with the normalization multiples) as

$$\mathcal{LM}_{y;Y}^{\alpha}\zeta_{\mathbf{Z}^{2}}(s;x+iy)=\frac{1}{\Gamma(\alpha)}\int_{0}^{\infty}\zeta_{\mathbf{Z}^{2}}(s;x+iyY)y^{\alpha-1}e^{-y}dy, \tag{1.2}$$

$$\mathcal{RL}_{y;Y}^{\alpha,\beta}\zeta_{\mathbf{Z}^{2}}(s;x+iy)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{0}^{1}\zeta_{\mathbf{Z}^{2}}(s;x+iyY)y^{\alpha-1}(1-y)^{\beta-1}dy \tag{1.3}$$

for $Y>0$. These can be regarded as weighted mean values of $\zeta_{\mathbf{Z}^{2}}(s;x+iy)$; the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integrals as $y\to+0$, while the functions e^{-y} and $(1-y)^{\beta-1}$ have effects to extract the parts corresponding to $y=O(Y)$ from $\zeta_{\mathbf{Z}^{2}}(s;z)$ with their respective weights. Note that the *confluence* operation

$$\mathcal{RL}_{y;\beta Y}^{\alpha,\beta}\zeta_{\mathbf{Z}^{2}}(s;x+iy)\xrightarrow[(\beta\to+\infty)]{}\mathcal{LM}_{y;Y}^{\alpha}\zeta_{\mathbf{Z}^{2}}(s;x+iy) \tag{1.4}$$

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is valid by the definitions (1.2) and (1.3), since $\zeta_{\mathbb{Z}^2}(s;x+iy) = O(y^{\max(0,1-2\sigma)})$ as $y \to +\infty$ (see Theorem 1 below).

It is of importance from both theoretical and applicational point of view to study asymptotic aspects of $\zeta_{\mathbb{Z}^2}(s;z)$ when $y = \operatorname{Im} z$ is large (cf. [CS1–CS2]). We have established in [Ka10] a complete asymptotic expansion of $\zeta_{\mathbb{Z}^2}(s;z)$ when $\operatorname{Im} z \to +\infty$, and that of the Laplace-Mellin transform (1.2) when $Y \to +\infty$. The subsequent paper [Ka11] proceeds to this direction by showing that a similar asymptotic series still exists for the Riemann-Liouville transform (1.3) when $Y \to +\infty$. It is the aim of this article to present these asymptotic expansions, together with their several consequences.

We first present a complete asymptotic expansion of $\zeta_{\mathbb{Z}^2}(s;z)$ when $\operatorname{Im} z \to +\infty$ (Theorem 1 below) upon giving an explicit (vertical) t-estimate for the remainder term. This theorem in particular clarifies the key ingredients by which the functional equataion of $\zeta_{\mathbb{Z}^2}(s;z)$ is to be valid (Corollary 1.1). Moreover, several specific cases of Theorem 1 naturally reduce to the Kronecker limit formula for $\zeta_{\mathbb{Z}^2}(s;z)$ when $s \to 1$, and to its variants for $\zeta_{\mathbb{Z}^2}(m;z)$ ($m = 2,3,\dots$) and $\zeta'_{\mathbb{Z}^2}(-n;z)$ ($n = 0,1,\dots$), where $\zeta'_{\mathbb{Z}^2}(s;z) = (\partial/\partial s)\zeta_{\mathbb{Z}^2}(s;z)$ (Corollaries 1.2 and 1.3). In connection with Theorem 1, Matsumoto [Ma] obtained asymptotic expansions (with respect to z) of holomorphic Eisenstein series, while Noda [No] derived an asymptotic formula (as $t \to +\infty$) for the non-holomorphic Eisenstein series on the line $\sigma = 1/2$. We next present complete asymptotoic expansions of the Laplace-Mellin transform (1.2) and of the Riemann-Liouville transform (1.3) both when $Y \to +\infty$ (Theorems 2 and 3 in Section 3). One can observe that the asymptotic expansion of (1.3) precisely reduces to that of (1.2) through the *confluence* operation (1.4). It should be noted that various hypergeometric functions appear and work in the proofs of these expansions; especially their summation and transformation properties play crucial rôles in the analysis of the remainder terms.

Prior to the proof of Theorem 1, we have prepared the analytic continuation of $\zeta_{\mathbb{Z}^2}(s;z)$ by means of Mellin-Barnes integral transformations (cf. [Ka10, Propositions 1 and 2]). This procedure was recently developed, independently of each other, by Kanemitsu-Tanigawa-Yoshimoto [KTY] (in a more general setting), and by the author [Ka10] for $\zeta_{\mathbb{Z}^2}(s;z)$; the procedure, differs slightly from other previously known method of the analytic continuation, gives a new alternative proof of the Fourier expansion of $\zeta_{\mathbb{Z}^2}(s;z)$, due to Chowla-Selberg [CS1–CS2]. We remark that Mellin-Barnes transformation technique was extensively utilized by Motohashi to investigate higher power moments of zeta and allied functions (see for e.g., [Mo1–Mo3]). The technique was also applied by the author [Ka1–Ka9] to study certain asymptotic aspects and transformation properties of zeta and theta functions.

2. Results on $\zeta_{\mathbb{Z}^2}(s;z)$

We write $\sigma_w(l) = \sum_{0<h|l} h^w$, and use the notations $e(z) = e^{2\pi i z}$ and

$$e^*(z) = e(z) + \overline{e(z)} = e(z) + e(-\overline{z}),$$

where $\overline{w}$ denotes the complex conjugate of w. We further introduce the function

$$\Phi^*_{r,s}(e(z)) = \sum_{h,k=1}^{\infty} h^r k^s e^*(hkz) = \sum_{l=1}^{\infty} \sigma_{r-s}(l) l^s e^*(lz),$$

which converges absolutely for all complex r, s if $\operatorname{Im} z > 0$, and for $\operatorname{Re} r < -1$, $\operatorname{Re} s < -1$ if $\operatorname{Im} z = 0$; in each case it defines a holomorphic function of r and s in the region of absolute convergence.

Let $\zeta(s)$ be the Riemann zeta-function, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol. Further let $U(\lambda;\nu;Z)$ denote the confluent hypergeometric function defined by

$$U(\lambda;\nu;Z) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-Zw} w^{\lambda-1}(1+w)^{\nu-\lambda-1} dw$$

for $\operatorname{Re}\lambda > 0$ and $|\arg Z| < \pi/2$ (cf. [Sl, p.5, 1.3]). Then our first main result asserts

Theorem 1. ([Ka10, Theorem 1]). *Let $\zeta_{Z^2}(s;z)$ be defined by* (1.1). *Then for any complex $z = x+iy$ with $y > 0$ and any integer $N \geq 0$ the formula*

$$\begin{aligned}\zeta_{Z^2}(s;z) &= 2\zeta(2s) + \frac{2\sqrt{\pi}\,\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-2s} \\ &+ \frac{2(2\pi)^{2s}}{\Gamma(s)}\{S_N(s,x;y) + R_N(s,x;y)\}\end{aligned}$$

holds in the region $-N < \sigma < 1+N$ except at $s=1$. Here

$$S_N(s;z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n (1-s)_n}{n!} \Phi^*_{s-n-1,-s-n}(e(z))(4\pi y)^{-s-n}$$

is the asymptotic series in the descending order of y, and R_N is the remainder term, which is expressed as

$$\begin{aligned}R_N(s;z) = &\frac{(-1)^N (s)_N (1-s)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^*(hkz) h^{2s-1} \\ &\times \int_0^1 \xi^{-s-N}(1-\xi)^{N-1} U(s+N;2s;4\pi hky/\xi) d\xi\end{aligned}$$

for $N \geq 0$ (the case $N=0$ should read without the factor $(-1)!$ and the ξ-integration), satisfying the estimate

$$R_N(s;z) = O\{(|t|+1)^{2N} e^{-2\pi y} y^{-\sigma-N}\}$$

for any $y \geq y_0 > 0$, in the region $-N < \sigma < 1+N$, where the O-constant depends on N, σ and y_0.

Remark. We see that

$$\Phi^*_{r,s}(e(z)) = e^*(z) + O\left\{\sum_{l=2}^{\infty} l^{\max(\mathrm{Re}\, r, \mathrm{Re}\, s)+\varepsilon} |e^*(lz)|\right\} = e^*(z) + O(e^{-4\pi y})$$

as $y \to +\infty$, and hence

$$\Phi^*_{r,s}(e(z)) \ll e^{-2\pi y} \qquad (y \geq y_0 > 0).$$

Therefore the term with the index n in $S_N(s;z)$ is estimated as $\ll (|t|+1)^{2n} e^{-2\pi y} y^{-\sigma-n}$; this shows that the presence of the bound above for $R_N(s;z)$ is reasonable.

Let $\zeta^*_{\mathbf{Z}^2}(s;z)$ be defined by

$$\zeta_{\mathbf{Z}^2}(s;z) = 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-2s} + 2\zeta^*_{\mathbf{Z}^2}(s;z).$$

Then the proof of Theorem 1 show that the following functional equation of $\zeta^*_{\mathbf{Z}^2}(s;z)$ reduces eventually to the simple property

$$\Phi^*_{r,s}(e(z)) = \Phi^*_{s,r}(e(z)).$$

Corollary 1.1. ([Ka10, Corollary 1.1]). *For any real x, y with $y > 0$ the functional equation*

$$(y/\pi)^s \Gamma(s) \zeta^*_{\mathbf{Z}^2}(s;z) = (y/\pi)^{1-s}\Gamma(1-s)\zeta^*_{\mathbf{Z}^2}(1-s;z)$$

follows, and this with the functional equation of $\zeta(s)$ implies that

$$(y/\pi)^s \Gamma(s) \zeta_{\mathbf{Z}^2}(s;z) = (y/\pi)^{1-s}\Gamma(1-s)\zeta_{\mathbf{Z}^2}(1-s;z).$$

We next state the Kronecker limit formula for $\zeta_{\mathbf{Z}^2}(s;z)$ and its variants. Let $\eta(z) = e(z/24)\prod_{n=1}^{\infty}(1-e(nz))$ be the Dedekind eta function, $\gamma_0 = -\Gamma'(1)$ Euler's constant, and B_n the n-th Bernoulli number (cf. [Er, p.35, 1.13(1)]). Then

Corollary 1.2. ([Ka10, Corollary 1.2]). *For any complex $z = x+iy$ with $y > 0$ the following formulae hold:*

$$\begin{aligned}\lim_{s\to 1}\left\{\zeta_{\mathbf{Z}^2}(s;z) - \frac{\pi/y}{s-1}\right\} &= \frac{\pi^2}{3} + \frac{2\pi}{y}\{\gamma_0 - \log(2y) + \Phi^*_{0,-1}(e(z))\} \\ &= \frac{2\pi}{y}\{\gamma_0 - \log(3y|\eta(z)|^2)\},\end{aligned}$$

and for any integer $m \geq 2$,

$$\begin{aligned}\zeta_{\mathbf{Z}^2}(m;z) = {} & \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{(2m)!} + \frac{2\pi(2m-1)!}{\{2^{m-1}(m-1)!\}^2}\zeta(2m-1)y^{1-2m} \\ & + \frac{(2\pi)^{2m}}{\{(m-1)!\}^2}\sum_{n=0}^{m-1}\binom{m-1}{n}(m+n-1)! \\ & \times \Phi^*_{m-n-1,-m-n}(e(z))(4\pi y)^{-m-n}.\end{aligned}$$

Corollary 1.3. ([Ka10, Corollary 1.3]). *Let* $\zeta'_{\mathbb{Z}^2}(s;z) = (\partial/\partial s)\zeta_{\mathbb{Z}^2}(s;z)$. *Then for any complex* $z = x+iy$ *with* $y > 0$ *the following formulae hold:*

$$\zeta'_{\mathbb{Z}^2}(0;z) = -2\log 2\pi + \frac{\pi y}{3} + 2\Phi^*_{-1,0}(e(z)) = -2\log(2\pi|\eta(z)|^2),$$

and for any integer $m \geq 1$,

$$\begin{aligned}\zeta'_{\mathbb{Z}^2}(-m;z) &= \frac{2(-1)^m(2m)!}{(2\pi)^{2m}}\zeta(2m+1) + \frac{2\pi(2^m m!)^2 B_{2m+2}}{(2m+1)!(m+1)}y^{2m+1} \\ &\quad + \frac{2(-1)^m}{(2\pi)^{2m}}\sum_{n=0}^{m}\binom{m}{n}(m+n)!\Phi^*_{-m-n-1,m-n}(e(z))(4\pi y)^{m-n}.\end{aligned}$$

3. Results on $\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;z)$ and $\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s;z)$

We write

$$\Gamma\binom{\alpha_1,\ldots,\alpha_m}{\beta_1,\ldots,\beta_n} = \frac{\prod_{h=1}^{m}\Gamma(\alpha_h)}{\prod_{k=1}^{n}\Gamma(\beta_k)}$$

for complex numbers α_h, β_k ($1 \leq h \leq m$; $1 \leq k \leq n$), and denote the generalized hypergeometric function by ${}_mF_n\left({\alpha_1,\ldots,\alpha_m \atop \beta_1,\ldots,\beta_n};z\right)$ for $m \leq n+1$. Then our second main result can be stated as

Theorem 2. ([Ka10, Theorem 2]). *Let* α *be fixed with* $\operatorname{Re}\alpha > 1$. *Then for any integer* $N \geq 0$ *and any real* x, Y *with* $Y > 0$ *the formula*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) &= 2\zeta(2s) + 2\sqrt{\pi}\Gamma\binom{s-1/2,\ \alpha+1-2s}{s,\ \alpha}\zeta(2s-1)Y^{1-2s} \\ &\quad + \frac{2\pi^{2s}}{\Gamma(s)}\{S_{\alpha,N}(s,x;Y) + R_{\alpha,N}(s,x;Y)\}\end{aligned}$$

holds in the region $\sigma < \operatorname{Re}\alpha/2$. *Here*

$$\begin{aligned}S_{\alpha,N}(s,x;Y) &= \sum_{n=0}^{N-1}\frac{(-1)^n(\alpha)_n}{n!}\Gamma\binom{(\alpha+n+1)/2-s}{(\alpha+n+1)/2} \\ &\quad \times \Phi^*_{2s-1-\alpha-n,-\alpha-n}(e(x))(2\pi Y)^{-\alpha-n}\end{aligned}$$

is the asymptotic series in the descending order of Y, *and* $R_{\alpha,N}$ *is the remainder term, which is expressed as*

$$\begin{aligned}R_{\alpha,N}(s,x;Y) &= \frac{(-1)^N}{(N-1)!}\Gamma\binom{\alpha+1-2s}{\alpha+1-s}\sum_{h,k=1}^{\infty}e^*(hkx)h^{2s-1} \\ &\quad \times \int_0^1 \xi^{-\alpha-N}(1-\xi)^{N-1}(1+2\pi hkY/\xi)^{-\alpha-N} \\ &\quad \times {}_2F_1\left({\alpha+N,\ s \atop \alpha+N+1-s};\frac{1-2\pi hkY/\xi}{1+2\pi hkY/\xi}\right)d\xi\end{aligned}$$

for $N \geq 0$ *(the case* $N = 0$ *should read without the factor* $(-1)!$ *and the* ξ*-integration), satisfying the estimate*

$$R_{\alpha,N}(s,x;Y) = O(Y^{-\operatorname{Re}\alpha - N})$$

for any $Y \geq Y_0 > 0$*, in the region* $\sigma < \operatorname{Re}\alpha/2$*, where the* O*-constant depends at most on* α*,* N*,* σ*,* t *and* Y_0*. In particular when* $\alpha \in \mathbb{R}$*, more explicitly*

$$R_{\alpha,N}(s,x;Y) = O\{e^{-\pi|t|/2}(|t|+1)^{(\alpha+N)/2-\sigma}Y^{-\alpha-N}\}$$

for any $Y \geq Y_0 > 0$ *in the region* $\sigma < \alpha/2$*, where the* O*-constant depends on* α*,* N *and* σ*.*

Remark 2.1. The condition $\operatorname{Re}\alpha > 1$ is crucial for the convergence of $\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy)$, especially for that of $\mathcal{LM}^{\alpha}_{y;Y}\zeta^*_{\mathbb{Z}^2}(s;x+iy)$.

Remark 2.2. It is seen that

$$|\Phi^*_{r,s}(e(x))| \leq 2\zeta(-\operatorname{Re} r)\zeta(-\operatorname{Re} s) < +\infty$$

for $\operatorname{Re} r < -1$, $\operatorname{Re} s < -1$, and hence when $\alpha \in \mathbb{R}$ the term with the index n in $S_{\alpha,N}(s,x;Y)$ is estimated as $\ll e^{-\pi|t|/2}(|t|+1)^{(\alpha+n)/2-\sigma}Y^{-\alpha-n}$; this shows that the presence of the bound for $R_{\alpha,N}(s,x;Y)$ above is reasonable.

It is in fact shown that $\lim_{N\to\infty} R_{\alpha,N}(s,x;Y) = 0$ for $\sigma < \operatorname{Re}\alpha/2$ and $Y > 1/2\pi$. The limiting case $N \to \infty$ of Theorem 2 therefore gives

Corollary 2.1. ([Ka10, Corollary 2.1]). *For any real* x*,* Y *with* $Y > 1/2\pi$ *the formula*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) &= 2\zeta(2s) + 2\sqrt{\pi}\Gamma\begin{pmatrix} s-1/2,\ \alpha+1-2s \\ s,\ \alpha \end{pmatrix}\zeta(2s-1)Y^{1-2s} \\ &\quad + \frac{2\pi^{2s}}{\Gamma(s)}S^*_{\alpha}(s,x;Y)\end{aligned}$$

holds in the region $\sigma < \operatorname{Re}\alpha/2$*, where*

$$\begin{aligned}S^*_{\alpha}(s,x;Y) = \sum_{n=0}^{\infty}&\frac{(-1)^n(\alpha)_n}{n!}\Gamma\begin{pmatrix}\alpha+n+1)/2-s \\ (\alpha+n+1)/2)\end{pmatrix} \\ &\times \Phi^*_{2s-1-\alpha-n,-\alpha-n}(e(x))(2\pi Y)^{-\alpha-n}.\end{aligned}$$

We next proceed to state our third main result.

Theorem 3. ([Ka11, Theorem 2]). *Let* α*,* β *be fixed with* $\operatorname{Re}\alpha > 1$*,* $\operatorname{Re}\beta > 1$*. Then for any integer* $N \geq 0$ *and any real* x*,* Y *with* $Y > 0$ *the formula*

$$\begin{aligned}\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy) &= 2\zeta(2s) + 2\sqrt{\pi}\Gamma\begin{pmatrix} s-1/2,\ \alpha+\beta,\ \alpha+1-2s \\ s,\ \alpha,\ \alpha+\beta+1-2s \end{pmatrix}\zeta(2s-1)Y^{1-2s} \\ &\quad + 2\pi^s\Gamma\begin{pmatrix}\alpha+\beta \\ s\end{pmatrix}\{S_{\alpha,\beta,N}(s,x;Y) + R_{\alpha,\beta,N}(s,x;Y)\}\end{aligned}$$

holds in the region $\sigma < \operatorname{Re}\alpha/2$. *Here*

$$S_{\alpha,\beta,N}(s,x;Y) = \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \Gamma\binom{(\alpha+n+1)/2-s}{(\alpha+n+1)/2,\ \beta-n} \times \Phi^*_{2s-1-\alpha-n,-\alpha-n}(e(x))(2\pi Y)^{-\alpha-n}$$

is the asymptotic series in the descending order of Y, *and* $R_{\alpha,\beta,N}$ *is the remainder term, which is expressed as*

$$R_{\alpha,\beta,N}(s,x;Y) = \frac{2^{2s}(-1)^N(\alpha)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^*(hkx)h^{2s-1} \times \int_0^1 \xi^{-\alpha-N}(1-\xi)^{N-1} F_{\alpha+N,\beta-N}(s;2\pi hkY/\xi)d\xi$$

for any $N \geq 0$ (*the case* $N = 0$ *should read without the factor* $(-1)!$ *and the* ξ-*integration*), *where*

$$F_{\alpha,\beta}(s;Z) = \Gamma\binom{1-2s}{1-s,\ \alpha+\beta}\, {}_2F_3\left(\begin{matrix}\alpha/2,\ (\alpha+1)/2\\(\alpha+\beta)/2,\ (\alpha+\beta+1)/2,\ s+1/2\end{matrix}; Z^2/4\right) + \Gamma\binom{2s-1,\ \alpha+1-2s}{s,\ \alpha,\ \alpha+\beta+1-2s}(2Z)^{1-2s} \times {}_2F_3\left(\begin{matrix}(\alpha+1)/2-s,\ \alpha/2+1-s\\(\alpha+\beta+1)/2-s,\ (\alpha+\beta)/2+1-s,\ 3/2-s\end{matrix}; Z^2/4\right)$$

with α, β *replaced by* $\alpha+n$, $\beta-N$, *and it satisfies*

$$R_{\alpha,\beta,N}(s,x;Y) = O(Y^{-\operatorname{Re}\alpha-N})$$

for any $Y \geq Y_0 > 0$ *in the region* $\sigma < \operatorname{Re}\alpha/2$, *where the* O-*constant depends on* α, β, N, σ, t *and* Y_0. *In particular when* $\alpha, \beta \in \mathbb{R}$, *more explicitly*

$$R_{\alpha,\beta,N}(s,x;Y) = O\{e^{-\pi|t|/2}(|t|+1)^{(\alpha+N)/2-\sigma}Y^{-\alpha-N}\}$$

for any $Y \geq Y_0 > 0$ *in the region* $\sigma < \alpha/2$, *where the* O-*constant depends on* α, β, N, σ *and* Y_0.

Corollary 3.1. ([Ka11, Corollary 2.1]). *The asymptotic expansion in Theorem 3 for* $\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy)$ *precisely reduces to that in Theorem 2 for* $\mathcal{LM}^{\alpha}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy)$ *through the confluence operation* (1.4).

Remark 3.1. The conditions $\operatorname{Re}\alpha > 1$ and $\operatorname{Re}\beta > 1$ are crucial for the convergence of $\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta_{\mathbb{Z}^2}(s;x+iy)$, especially for that of $\mathcal{RL}^{\alpha,\beta}_{y;Y}\zeta^*_{\mathbb{Z}^2}(s;x+iy)$.

Remark 3.2. Similarly to Remark 2.2, when $\alpha, \beta \in \mathbb{R}$ the term with the index n in $S_{\alpha,\beta,N}$ is estimated as $\ll e^{-\pi|t|/2}(|t|+1)^{(\alpha+n)/2-\sigma}Y^{-\alpha-n}$; this shows that the presece of the bound for $R_{\alpha,\beta,N}(s,x;Y)$ above is reasonable.

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